Infinite Exponential Function

Pedro Fernando Morales Almazán

pedro_morales@baylor.edu

Abstract

This article is about a strange function made taking a number and calculating infinitely many powers of it. At the begining, one can obtain some interesting results, but in the end, there is a extrange result that does not have a direct explication of what is happening. In this paper, I will try to explore this function and discuss about it's properties.

A few years ago, someone show me this interesting problem during our Real Analysis course:

Fing x such that

$$x^{x^{x^{\cdots}}} = 2$$

At the very begining, one could suspect that the answer is $\sqrt{2}$, so this was my actual first guess.

To get the solution, the problem can be rewritten as follows:

Find x such that the sequence $x_0 = x, \quad x_n = x^{x_{n-1}}$ has

$$\lim_{n\to\infty} x_n = 2$$

We have that $x \neq 0$ for the sequence be defined. Suppose that $x > 0$, we will consider the $x < 0$ case later. Then

$$2 = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x^{x_n}$$

and because $x > 0$, $x_n > 0, \forall n \in \mathbb{N}$ and together with the continuity of ln,

$$\ln 2 = \lim_{n\to\infty} \ln\left(x^{x_n}\right) = \lim_{n\to\infty} x_n \ln x = \ln x \lim_{n\to\infty} x_n = 2\ln x = \ln x^2$$

hence, $x^2 = 2$, and since x is positive, $x = \sqrt{2}$.

Once we solve this interesting problem, it arise the next question: *If it works for 2, does it works for $n \in \mathbb{N}$? for $r \in \mathbb{R}^+$?*

Then, suppose the same problem but changing the 2 with $r \in \mathbb{R}^+$

Find x such that

$$x^{x^{x^{x^{\cdots}}}} = r$$

hence, we can make a similar statement as the one used before, letting $x_n = x^{x_{n-1}}$ such that

$$\lim_{n\to\infty} x_n = r$$

then been $x > 0$, ln is defined and continous so

$$\ln r = \lim_{n\to\infty} \ln\left(x^{x_n}\right) = \lim_{n\to\infty} x_n \ln x = \ln x \lim_{n\to\infty} x_n = r \ln x = \ln x^r$$

and therefore the relation $r = x^r$ holds, so

$$x = r^{1/r}$$

Then, by this relation, we can find x such that the limit of the sequence is r, but an interesting fact comes when making $r = 4$. By our last result, this gives that $x = 4^{1/4}$, but this means that $x = 2^{1/2}$ and hence we have that

$$x^{x^{x^{x^{\cdots}}}} = 4$$

but in the other hand, by the initial result,

$$x^{x^{x^{x^{\cdots}}}} = 2$$

then, unless this is a clever way of proving that $2 = 4$, something wrong is happening here.

Trying to find the missing gap of this proof, this reminded me one IMO problem that used a similar fact to achive the desired result. At some point, it was needed to find all pairs (a, b) of distinct integers such that

$$a^b = b^a$$

it is not so hard to find a pair that satisfies the given conditions, one easily can find that $(2, 4)$ is a siutable pair, but the hard thing is to prove that actually this is the only pair that satisfies the equality. A good method to achive this is translating the problem into finding all pairs satisfing

$$a^{1/a} = b^{1/b}$$

wich is equivalent to the previous statement.

Defining f, $f : \mathbb{R}^+ \to \mathbb{R}^+$, by

$$f(r) = r^{1/r}$$

we have that $f(2) = f(4) = \sqrt{2}$. To analize the behavior of this function, we can find whenever the function is increasing and decreasing, and by this way stablish that there are no other answers besides 2 and 4.

Calucating f' we have

$$f'(r) = \frac{d}{dr}r^{1/r} = \frac{d}{dr}e^{(1/r)\ln r} = e^{(1/r)\ln r}\frac{d}{dr}(1/r)\ln r$$

$$= r^{1/r}\left(-\frac{\ln r}{r^2} + \frac{1}{r^2}\right) = \frac{r^{1/r}}{r^2}(1 - \ln r) = r^{\frac{1-2r}{r}}(1 - \ln r)$$

and now we have that the term $r^{\frac{1-2r}{r}}$ do not affect the sign of f', because it is always positive, so what determines it's sign is $1 - \ln r$, hence f is decreasent if $r > e$, increasent if $r < e$ and attains it's maximun at $r = e$. This means that the only way to get an equality of the type $f(a) = f(b)$ is that, without loss of generality, $a \in (0, e)$ and $b \in (e, \infty)$, so, the only avialable choices for a are 1 and 2. We can discart $a = 1$, because there is no $b \in (e, \infty)$ satisfing $f(b) = 1$, because in fact, 1 is and horizontal asymptote for f. This give us that the only choice left for a is 2 and hence $b = 4$ and there are no other a and b that satisfy the given relationship.

What this means is that if we put $g : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$g(x) = x^{x^{x^{x^{\cdots}}}}$$

$f \circ g(x) = x$ wich means that $g = f^{-1}$. The problem before was that f^{-1} is not actually a function, because f is not one-to-one, and hence we can not actually take $g \circ f$ as a function, because g is not a funcion while been defined as f^{-1}. This behavior is the same as the one in $\sin x, \arcsin x$ that we only take a *branch* of $\arcsin x$ instead of the strictly spoken $\sin^{-1} x$.